BE/APh 161: Physical Biology of the Cell, Winter 2025
Homework #10
Due 2:30 PM, March 19, 2025.

Problem 9.1 (Equation of motion for the cortex, 60 pts)**.**
In Chapter 14 of their recent book on active matter, *The Restless Cell*, Phillips and Hueschen give the following equation to describe the motion of the cell cortex.

$$-\eta\,\partial_x^2 v_x + \gamma\, v_x = \partial_x C. \tag{9.1}$$

Note that when we have repeated indices that are x, y, or z, summation is *not* assumed. In the notation we have been using in lecture, this is

$$-\eta\,\partial_x^2 v_x + \gamma\, v_x = \partial_x \sigma_a. \tag{9.2}$$

This equation describes how a gradient in active stress drives cortical flow against viscous dissipation and frictional losses. In this problem, you will derive this equation. Even though the nematic order does not appear in the equation, it is necessary in its derivation. We saw in lecture that we must have anisotropy to be able to support active stresses; this will become clear when we consider the nematic order explicitly in deriving the above cortical equation of motion.

In lecture, we defined the nematic order parameter as

$$Q_{ij} = S\left(n_i n_j - \frac{1}{3}\,\delta_{ij}\right). \tag{9.3}$$

Here, S is the magnitude of the local order. We wrote the active stress as a Taylor series expansion of the nematic order parameter as

$$\sigma_\text{active} = \sigma_a^0\,\delta_{ij} + \sigma_a Q_{ij}. \tag{9.4}$$

Then, the stress tensor for a three-dimensional active nematic viscous fluid, which is how are are modeling the cortex, is

$$\sigma_{ij} = -\Pi\,\delta_{ij} + 2\eta\, v_{ij} + \sigma_{ij}^\text{nematic} + \sigma_a Q_{ij}, \tag{9.5}$$

where $\Pi = p - \sigma_a^0$ and v_{ij} is the symmetric part of the velocity gradient tensor,

$$v_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right). \tag{9.6}$$

We denote by $\sigma_{ij}^\text{nematic}$ the passive stresses due to nematic order. The $\sigma_a Q_{ij}$ term is directional active stress exerted along the nematic order. The equation of motion is then, considering again the interialess limit for an incompressible fluid,

$$\partial_j \sigma_{ij} = 0 = -\partial_i \Pi + \eta\,\partial_j\partial_j v_i + \partial_j \sigma_{ij}^\text{nematic} + \partial_j(\sigma_a Q_{ij}). \tag{9.7}$$

Starting from these equations, you will derive the equation of motion for the cortex, (9.2).

a) To simplify things, we will assume that the alignment of filaments in the cortex rapidly relax to equilibrium so that the dynamics of the order parameter may be neglected, i.e., Q_{ij} is constant. To find the equilibrium value of Q_{ij}, we note that the deformation energy of a nematic liquid crystal can be approximately written as

$$F_d = F_0 + \frac{\chi}{2} Q_{ij}Q_{ij} + \frac{L}{2} \left(\partial_k Q_{ij}\right) \left(\partial_k Q_{ij}\right). \tag{9.8}$$

Give an explanation as to why this is a reasonable functional form for the deformation free energy.[1]

We can find $\sigma_{ij}^{\text{nematic}}$ to be related to the functional derivative of the deformation energy. You do not need to derive this, but you get the result we stated in lecture.

$$\sigma_{ij}^{\text{nematic}} = \beta_1 \left(\chi - L\partial_k\partial_k\right) Q_{ij}. \tag{9.9}$$

b) The cortex is essentially a two-dimensional object. It is only about one micron thick, but has extent of over 50 microns. We therefore assume that the filaments of the cortex are aligned only in the x-y plane. In other words, $n_z = 0$, which means that $Q_{xz} = Q_{yz} = 0$ and $Q_{zz} = -S/3$. Given that it is constrained to two dimensions, find the value of Q_{ij} that minimizes the deformation free energy, subject to the constraint that alignment is confined to a thin sheet, i.e., that $n_z \approx 0$. Your result will be linear in S.

c) Next, we specify that the cortex does not bend or buckle, so the stresses normal to the two-dimensional cortex should vanish. In other words, $\sigma_{zz} = 0$. Based on this assumption, derive an expression for Π. *Hint:* Don't forget that in three dimensions, the material is incompressible, so $\partial_i v_i = 0$.

d) Using the expression you derived in part (c), along with the assumption that Q_{ij} is constant, show that the two dimensional equations of motion are

$$\eta\,\partial_z^2 v_x + 3\eta\,\partial_x^2 v_x + \eta\left(\partial_y^2 v_x + 2\partial_x\partial_y v_y\right) + \partial_x \sigma_a = 0, \tag{9.10}$$

$$\eta\,\partial_z^2 v_y + 3\eta\,\partial_y^2 v_y + \eta\left(\partial_x^2 v_y + 2\partial_y\partial_x v_x\right) + \partial_y \sigma_a = 0, \tag{9.11}$$

where we have absorbed a factor of $S/2$ into σ_a.

e) Show that

$$\eta\,\partial_z^2 v_x = \partial_z \sigma_{xz} + \eta\,\partial_x(\partial_x v_x + \partial_y v_y). \tag{9.12}$$

[1] There are also deep arguments about symmetry that come into play here, but you do not need to worry too much about those. You can read more about this particular form of the free energy; it is called a Landau-de Gennes expansion.

A similar relation holds in the y-direction. As a result, we have

$$\partial_z \sigma_{xz} + \eta\,(\partial_x^2 + \partial_y^2) v_x + 3\eta\,\partial_x(\partial_x v_x + \partial_y v_y) + \partial_x \sigma_a = 0, \tag{9.13}$$

$$\partial_z \sigma_{yz} + \eta\,(\partial_x^2 + \partial_y^2) v_y + 3\eta\,\partial_y(\partial_x v_x + \partial_y v_y) + \partial_y \sigma_a = 0. \tag{9.14}$$

f) Next, we will average the two equations over the thin dimension, z. That is, we will apply the operation $h^{-1}\int_0^h \mathrm{d}z$, where h is the cortical thickness, to each equation. We define

$$\bar{a} = \frac{1}{h}\int_0^h \mathrm{d}z\, a, \tag{9.15}$$

where a is some physical quantity, such as v_x. Show that if $\partial_x h \approx 0$, i.e., if h is approximately constant, then $\overline{\partial_x a} \approx \partial_x \bar{a}$. Going forward, you may assume that similar results hold for $\overline{\partial_y a}$, $\overline{\partial_x \partial_y a}$, and so on.

g) Perform the averages over equations (9.13) and (9.14). You will be left with a term like $h^{-1}\,\sigma_{xz}\big|_0^h$. Explain why we can write

$$\frac{1}{h}\;\sigma_{xz}\big|_0^h = -\gamma\,\bar{v}_x. \tag{9.16}$$

What is the meaning of the parameter γ?

h) Your equations should now look like

$$-\gamma\,\bar{v}_x + \eta\,(\partial_x^2 + \partial_y^2)\bar{v}_x + 3\eta\,\partial_x(\partial_x \bar{v}_x + \partial_y \bar{v}_y) + \partial_x \bar{\sigma}_a = 0, \tag{9.17}$$

$$-\gamma\,\bar{v}_y + \eta\,(\partial_x^2 + \partial_y^2)\bar{v}_y + 3\eta\,\partial_y(\partial_x \bar{v}_x + \partial_y \bar{v}_y) + \partial_y \bar{\sigma}_a = 0. \tag{9.18}$$

Explain why the quantity 3η is a two-dimensional bulk viscosity.

i) Now, we will assume that we can neglect curvature and that we have azimuthal symmetry in the *C. elegans* cortex. Under these assumptions, write a simplified version of equation (9.17). Is equation (9.18) still necessary? Finally, what do we need to do to get the final result we are after, equation (9.2)?

j) Consider now a domain $0 \le x \le L$. Let σ_a^0 be the magnitude of the active stress. Identify a length scale ℓ such that you can nondimensionalize equation (9.2) to give

$$\partial_{\tilde{x}}\,\tilde{\sigma}_a = -\partial_{\tilde{x}}^2 \tilde{v} + \tilde{v}, \tag{9.19}$$

where tildes denote dimensionless quantities and the domain is now $0 \le \tilde{x} \le \tilde{L}$ with $\tilde{L} = L/\ell$. Provide an interpretation of the length scale ℓ.

k) Imagine we now have a very sharp active stress gradient at $\tilde{x} = \tilde{x}_0$, $\tilde{\sigma}_a = \theta(\tilde{x} - \tilde{x}_0)$, where $\theta(x)$ denotes the Heaviside step function. It can be shown that when material cannot flow through the ends ($\tilde{v}(0) = \tilde{v}(\tilde{L}) = 0$),

$$\tilde{v}(\tilde{x}) = \frac{\sinh(\tilde{L} - \tilde{x}_0)}{\sinh(\tilde{L})} \sinh(\tilde{x}) - \theta(\tilde{x} - \tilde{x}_0) \sinh(\tilde{x} - \tilde{x}_0). \tag{9.20}$$

(Can you derive this?) Plot this result for $\tilde{x}_0 = 7\tilde{L}/10$ for various values of $\tilde{L}$. Can a local active stress gradient drive flow over long length scales?

Problem 9.2 (Optical cell stretching, 70 pts).
We briefly discussed optical cell stretchers in lecture. Optical cell stretchers work by taking advantage of the difference in index of refraction between a cell and the surrounding solution to trap a free cell in two counter-propagating laser beams. The power of the laser is then increased to exert stress and elongate the trapped cell. The induced stress is proportional to the laser power. The constant of proportionality, F_G is dependent on geometry and cannot be ascertained. The deformation (strain) is measured by taking images with a light microscope. The process is illustrated in Figure 1. In this way, the mechanical properties of an entire cell can be measured.

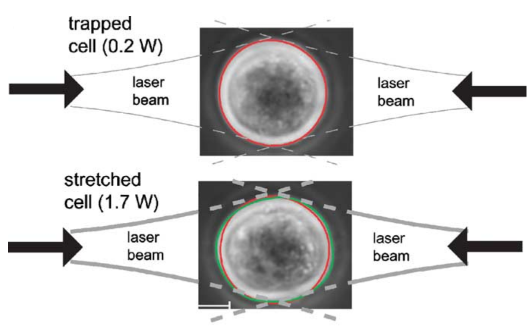

Figure 1: Schematic of an optical stretcher. The cell stretches along the axis parallel to the laser beams. The strain is given by the fractional change of the diameter of the cell along the stretching axis. Figure take from Wottawah, et al., *Acta Biomaterialia*, **1**, 263–271, 2005.

This technique was used to assess the mechanical properties of two mammalian cell types, 3T3 and SVT2 (which have reduced actin), in Wottawah, et al., *PRL*, **94**, 098103, 2005. In this work, the authors performed a stress step experiment in which a constant stress σ_0 was applied at $t = 0$, as in lecture. The stress was set back to zero at time $t = t_1$. The authors can obtain the creep compliance from this measurement.

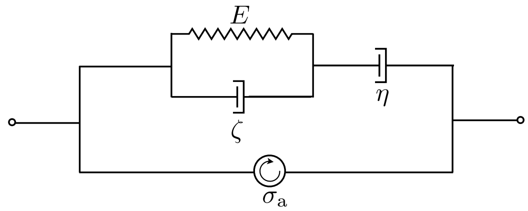

Figure 2: Schematic of an active Jeffreys fluid.

a) Derive an expression for the strain in the stress step experiment if we model the cell as an active Jeffreys fluid as in Figure 2. The stress step can be described mathematically as

$$\sigma(t) = F_G\, \sigma_0\, \theta(t)\, \theta(t_1 - t), \tag{9.21}$$

where $\theta(t)$ is the Heaviside step function. Assume the active stress is constant, given by σ_a.

b) The authors perform curve fits of the expression you derived in part (a) to get values for the parameters of the cell. Explain why they cannot independently measure E, η, and ζ, but only products thereof. Can a constant active stress be detected in this experiment?

c) The authors then use the curve fit parameters to compute the storage and loss moduli (E' and E'') of the cell. Derive expressions for the storage and loss moduli from the fit parameters. (*Note*: These reported storage and loss moduli are dependent on choosing a model for the viscoelastic behavior of the cell. This is not ideal, but is apparently a necessity due to experimental constraints.)